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Optimal advertising for a generalized Vidale–Wolfe response model

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Abstract

In this research, we formulate budget allocation decisions as an optimal control problem using a generalized Vidale–Wolfe model (GVW) as its advertising dynamics under a finite time horizon. One key element of our modeling work is that the proposed optimal budget allocation model (called GVW-OB) takes into account the roles of two useful indexes of the GVW model representing the advertising elasticity and the word-of-mouth (WoM) effect, respectively, in determining optimal budget. Moreover, we discuss desirable properties and provide a feasible solution to our GVW-OB model. We conduct computational experiments to assess our model's performance and its identified properties, based on real-world datasets obtained from advertising campaigns by three e-commerce companies on Google AdWords, Facebook Ads and Baidu Ads, respectively. Experimental results show that (1) our GVW-OB strategy outperforms four baselines in terms of both payoff and ROI in either concave or S-shaped settings; (2) linear budget allocation strategies favor concave advertising responses, while nonlinear strategies support S-shaped responses; (3) a larger ad elasticity empowers higher levels of optimal budget and corresponding market share and thus achieves higher payoff and ROI, so does a larger WoM effect; and (4) as the total budget increases, the resulting payoff by the GVW-OB strategy increases monotonically, but the ROI decreases, which is consistent with the law of diminishing marginal utility. From a methodological perspective, our GVW-OB strategy provides a feasible solution for advertisers to make optimal budget allocation over time, which can be easily applied to a variety of advertising media. The identified properties and experimental findings of this research illuminate critical managerial insights for advertisers and media providers.

Keywords Advertising models · Budget decisions · Optimal control · Vidale–Wolfe model

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1 Introduction

The Internet has witnessed the advent of a large number of digital media vehicles (e.g., search portals, social media platforms, e-commerce platforms, online gaming, mobile apps, online videos, banners, etc.) promising a variety of novel advertising forms [76]. According to Interactive Advertising Bureau [25], advertisers are dispensing a significant proportion of their advertising dollars (i.e., \$88.0 billion in the U.S. alone) toward digital advertising in 2017, a 21.4% increase over F.Y. 2016. Digital media vehicles and related advertising forms offer rich means to reach potential consumers. However, digital advertising environments have evolved into a vastly complex communication system [67, 74]. In the wake of this development, the allocation of advertising resources becomes a critical issue for advertisers [71].

From an operational standpoint, determining the optimal budget level is the first and foremost decision problem faced by advertisers, which heavily affects other advertising decisions [27, 78]. There are many challenges associated with budget allocation decisions in digital advertising channels (e.g., search ads and social media ads). First, it requires a comprehensive understanding of the process of advertising responses in digital media. Several classical advertising response models and their derivatives have been extensively examined (e.g., [14, 40, 52, 64, 70] for traditional advertising. However, advertising responses in digital advertising media are more complicated in terms of underlying mechanisms and processes, thus, need more sophisticated models [79]. Second, typically, digital advertising markets are incredibly dynamic, owing to the vibrant nature of the underlying mechanism and consumers' behaviors [28, 38, 81]. However, advertisers usually have minimal sufficient knowledge and time to handle advertising decisions in such dynamic environments. Third, it is not always the case that spending more on advertising leads to higher profits [75].

The objective of this study is to explore budget allocation decisions based on a generalized Vidale–Wolfe (GVW) model with applications to digital advertising. In this research, we formulate budget allocation decisions as an optimal control problem using the GVW model as its advertising dynamics, under a finite time horizon. First, one key element of our modeling work is that the proposed optimal budget allocation model (called GVW-OB) takes into account roles of two useful indexes of the GVW model representing the advertising elasticity and the word-of-mouth (WoM) effect, respectively, in determining optimal budget. As observed by prior studies, the ad elasticity (e.g., [23, 41]) and the word-of-mouth effect (e.g., [1, 29, 58]) are highly prominent in digital advertising environments, especially in e-commerce markets. Second, we discuss some desirable analytical properties and provide a computational solution to the GVW-OB model. Furthermore, based on real-world datasets obtained from advertising campaigns by three e-commerce companies on Google AdWords, Facebook Ads and Baidu Ads, respectively, we conduct computational experiments to validate our model and its identified properties.

Experimental results show that our GVW-OB strategy outperforms four baselines in terms of both payoff and ROI in either concave or S-shaped settings. This proves the potential value of the ad elasticity index and the WoM index for optimal advertising decisions. Moreover, linear budget allocation strategies favor the concave advertising responses, while nonlinear strategies support the S-shaped responses. The ad elasticity index and the WoM index have significant influences on the performance of the GVW-OB strategy, in terms of resulting payoff and ROI. More specifically, a larger ad elasticity empowers higher levels of optimal budget and corresponding market share and thus achieves higher payoff and ROI, so does a larger WoM effect. In a nutshell, the nonlinearity modeling feature with respect to advertising budget (by the ad elasticity index) and the untapped market share (by the WoM index) improves the adaptability of budget allocation strategies to handling complex advertising situations that specifically characterize digital media. Experiments on sensitivity analysis with respect to the ad elasticity index and the WoM index also reveal that, as the total budget increases, the resulting payoff by the GVW-OB strategy increases monotonically, but the ROI decreases. This is consistent with the law of diminishing marginal utility, which proves an attractive feature of our GVW-OB strategy. From a methodological perspective, our GVW-OB strategy provides a feasible solution for advertisers to make optimal budget allocation over time, which can be easily applied to a variety of advertising media. The identified properties and experimental findings of this research illuminate critical managerial insights for advertisers in various advertising forms.

The contributions of this work can be summarized as follows. Overall this research contributes to the literature on optimal advertising budget allocation. First, we investigate an optimal control problem using the GVW model as its advertising dynamics under a finite time horizon, and derive desirable properties of our proposed GVW-OB model through mathematical analysis. As far as we knew, this is the first effort to formally make phase diagram analysis of the optimal budget allocation model with respect to the ad elasticity index and the WoM index. Second, compared to the VW model and its derivatives, the GVW model is more nonlinear with respect to both the decision variable (i.e., advertising budget) and the state variable (i.e., market share). This feature entitles the GVW-OB model to efficiently handle budget allocation problems in complex advertising environments; meanwhile, it makes the optimal solution analytically intractable. This research provides a closed-loop solution for the GVW-OB model in situations with and without budget constraints. Methodologically, the present study gives advertisers a feasible approach to make budget decisions in real-time while managing their advertising campaigns. Third, we conduct computational experiments to validate the GVW-OB strategy and its identified properties with real-world data. Specifically, we compare the GVW-OB strategy with baseline strategies based on the VW model and its derivatives, and examine effects of the ad elasticity and the WoM indexes on optimal budget path and corresponding evolution of market share. The outcome of this research enhances understanding of advertising responses and especially the role of the ad elasticity and the WoM indexes.

The remainder of this paper is organized as follows. The next section provides a literature review on advertising response models and optimal advertising budget allocation. Section 3 presents an overview of the GVW model and based on it builds an optimal budget allocation model. Section 4 discusses desirable properties and provides a feasible solution for the GVW-OB model. Section 5 reports experimental results to validate our GVW-OB strategy and its identified properties. Section 6 discusses normative findings and managerial insights from this work as well as its shortcomings. Finally, we conclude this work in Sect. 7.

2 Related work

In this section, we first survey the literature on advertising response models, and then focus on optimal advertising budget allocation.

2.1 Advertising response models

In advertising literature, it typically takes response models to describe the dynamic advertising-sales process. Advertising response models parsimoniously describe the relationship between advertising budget (or expenditure) and unit sales [55, 60]. The well-known work of Vidale and Wolfe [70] took the initiative to define the concept of advertising effectiveness and developed a differential equation to capture advertising-sales response dynamics. Another pioneering work by Nerlove and Arrow [52] invented a goodwill accumulation equation to trace the carryover effect of current advertising campaigns on consumers' future purchasing behaviors. This study takes a derivative of the VW model as advertising response dynamics to explore the optimal budget allocation problem. Our motivation for choosing VW-type models lays on the fact that they combine several important market factors that influence advertising budget decisions, such as the carryover effect of past advertising on current sales [7], the saturation level [36], the possible diminishing returns to cumulative advertising budget [14, 40, 82], in a time-varying manner. Thus, in the following we mainly pay attention to VW-type advertising response models.

Technically, the VW model cannot describe competitive effects in the situation with two or more advertisers. A variation of the Lanchester model was introduced by Kimball [34] to analyze competitive advertising decisions, which can also be viewed as the precursor of the third genre of response functions. As noted by Little [40], the Lanchester model is essentially the generalization of the VW model from a competitive perspective, which has been widely extended to study competitive situations where each firm uses advertising to grab market share of her rivals [7, 31, 43].

In the past decades, plenty of research efforts in this direction have been invested in extending the VW response model to various complicated situations. The first research stream is related to the ad elasticity which determines the relationship between advertising budget and advertising effort. Conceptually, the ad elasticity describes the effectiveness of current advertising budget in the current and future periods [23, 35, 41]. In the VW model, advertising effort is represented as a linear form, i.e., $f(u) = u$, where u represents advertising budget. In other words, a uniform return can be obtained from each advertising dollar. However, the linear assumption is unrealistic or unattainable in advertising practices because it opposes the law of diminishing marginal utility, i.e., the marginal return decreases as the investment increases, holding other factors constant [39]. To this end, prior studies (e.g., [44, 45, 47, 48, 50, 82]) have investigated desirable forms of advertising effort with respect to advertising budget. As suggested by Little [40], Mesak [46] and Mesak and Zhang [49] employed an explicit form $f(u) = u^{\alpha}$ (where α stands for the ad elasticity) to analyze various pulsing responses and solved an advertising pulsation problem for a monopolistic firm. The exponential form has been extensively adopted for advertising research in monopolistic and competitive settings (e.g., [12, 14, 77, 78]).

The second research stream on the VW model extensions is concerned with the word-of-mouth (WoM) effect, representing the additional effect generated from advertising effort through communications between individuals comprising the sold portion and those comprising the unsold portion of the potential market [33, 64, 66, 80]. Sethi [64] incorporated the WoM effect into the VW model by using a square-root form of the untapped potential (i.e., $\sqrt{1-x}$, where x denotes the sold portion), which was echoed by the research by Sorger [66] on advertising competition. The positive effect $\rho u\sqrt{1-x}$ can be approximated by $\rho u(1-x) + \rho ux(1-x)$ where ρ denotes the response constant, of which the additional term $\rho ux(1-x)$ can be explained by the process of WoM communications between individuals who are already aware of the product and who are not, with ρu representing the intensity of communications at a certain advertising level. Yang et al. [73] presented a generalized form of the GVW model representing the ad elasticity and the WoM effect by two additional indexes (i.e., the ad elasticity index and the WoM index), and developed a deep neural network (DNN)-based method to learn modeling parameters.

The third research stream is with the incorporation of marketing-mix variables. Together with advertising budget, marketing-mix variables such as product quality and price also play key roles in generating sales and maximizing profit of a firm. Mathmatically, advertising budget is taken as the only marketing variable to capture the relationship between advertising budget and sales (or market share) in the VW model. In the literature, prior research (e.g., [56, 57, 62]) has devoted to incorporating other marketing-mix variables (z) into the VW model, i.e., $f(u, z)$.

Last but not least, stochastic extensions of the VW model have been developed to study stochastic budget allocation problems in uncertain markets where several variables for response functions cannot be exactly known in advance [15, 24, 54].

2.2 Optimal budget allocation

Optimal budget allocation has attracted increasing research interests in exploring solutions for advertising decisions, based on different variations of advertising response models, either optimal allocation for an individual advertiser [19, 37, 61, 75] or equilibrium solutions in a competitive setting [11, 31, 53, 72]. As far as we knew from the literature, most research efforts on advertising competition typically employed a competitive variation of the VW model (i.e., the Lanchester model). Since advertising environments are essentially time-varying, optimal control methods (for details see [65, 77]) and differential games (for details see [8, 10, 26, 32]) are commonly utilized to develop optimal advertising strategies.

Owing to its simplicity, the VW advertising model enables the development of analytical solutions for budget allocation problems. For example, Sethi [63] dealt with an optimal control problem for the VW model to maximize the present value of total profit under a finite horizon. Specifically, optimal solution for the budget allocation problem based on the VW model is a form of bang-bang control due to its linearity with respect to advertising budget. For more detailed information about the bang-bang solution, please refer to Sethi and Thompson [65].

As for the VW extensions with respect to the ad elasticity, typically researchers took an exponential form of advertising effort with the ad elasticity (α) as 1/2 (e.g., [5, 59, 72]). Note that budget allocation models with the quadratic form of the decision variable (i.e., advertising budget) in the objective function and the VW model or the Lanchester model as the state equation (e.g., [20, 21, 43]) also fall into this category. These models are equivalent to those based on the VW extensions with $\alpha = 1/2$. Chintagunta and Vilcassim [5] used the Lanchester model with $\alpha = 1/2$ and developed open-loop and closed-loop strategies to determine equilibrium advertising levels for two advertising rivals. Fruchter and Kalish [20] obtained analytical solutions of a differential game where the objective functions contain the quadratic form of advertising budget (u^2) in a duopolistic market. Later on, Fruchter [21] extended this model for an oligopolistic market. Ezimadu and Nwozo [15] considered cooperative advertising in a manufacturer–retailer supply chain in a modeling framework with u^2 in the objective function, and obtained optimal strategies for the manufacturer and the retailer.

Regarding the VW extensions with respect to the WoM effect, Sethi [64] explored both deterministic and stochastic optimal dynamic advertising problems based on a variation of the VW model with a square-root form of the untapped market (i.e., $\sqrt{1 - x}$). Yang et al. [77] proposed a solution framework to derive the optimal budget allocation strategy across several search markets, using an extended Sethi model to better suit the sponsored search advertising scenarios by incorporating the dynamic advertising effort and quality score. Sorger [66] studied the open-loop and closed-loop Nash equilibrium outcomes in a duopolistic market with a variation of the Lanchester model with $\sqrt{1 - x}$.

In the branch on interfacing with other marketing-mix variables, Sethi et al. [62] solved an optimal control problem incorporating price and advertising effects over time. Reddy et al. [56] integrated design quality as a control variable in the VW model and determined optimal advertising and quality investments using the impulse optimal control.

By assuming that advertising budget affects the probability of sales, Tapiero [68] addressed adaptive advertising control problems with a probabilistic version of the VW model. Prasad and Sethi [54] investigated the optimal budget allocation problem in a duopolistic market based on a stochastic, competitive version of the VW model with a churn term, and derived explicit solutions and comparative statics for closed-loop Nash equilibria for symmetric as well as asymmetric competitors.

Comparing with the literature on optimal advertising budget allocation discussed above, our research is different in the following aspects. First, this research takes the GVW model to explore the optimal budget allocation strategy as its advertising dynamics. The GVW model could flexibly capture nonlinear advertising responses in practice and learn modeling parameters over time [73]. In this sense, the GVW-OB strategy can take those advantages to enhance its applicability in dynamic and complex advertising environments. However, the GVW model also raises a big challenge for obtaining the optimal solution because it makes the optimal budget path analytically intractable, which serves as the motivation for the present study. Note that, the focus of this paper differs from that of Yang et al. [73]. Specifically, Yang et al. [73] focused on developing the GVW model by analyzing its modeling properties and the estimation method, while this paper is concerned with the optimal budget allocation strategy using the GVW model as its advertising dynamics. Second, we analytically derive properties of the GVW-OB model, and develop a closed-loop solution to compute the optimal budget path and corresponding evolution of market share. Third, computational experiments were conducted to validate the proposed GVW-OB strategy and identified properties by applying a 2×3 design with two shapes of advertising responses (i.e., concave and S-shaped) and three real-world datasets (i.e., a U.S. firm's advertising campaigns on Google AdWords, a European firm's advertising campaigns on Facebook Ads and a Chinese firm's advertising campaigns on Baidu Ads). In addition, this research makes sensitivity analysis to examine influences of the ad elasticity index and the WoM index on the performance of the GVW-OB strategy.

3 The optimal budget allocation model

In this section, we present a formal model of budget allocation (GVW-OB) to maximize the expected profit, with the GVW model describing the advertising response dynamics. Moreover, we also consider the two types of budgeting situations for advertisers, i.e., they have either a limited or a sufficient budget, which defines the feasible decision space and thus, to a major degree, determines the optimal trajectory of the optimal advertising budget and corresponding market share. We will explore this in more details in Sect. 4.

3.1 Advertising response model

The advertising response model provides a formal way to represent the accumulative effect obtained from advertising campaigns. The work of Vidale and Wolfe [70] is one of the pioneering and earliest works in this direction. To fit flexible decision scenarios in various advertising forms, this research employs the GVW model (for more details, refer to [73]), which is given as follows.

$$\dot{x} = \rho u^{\alpha}(1 - x)^{\beta} - \delta x, x(0) = x_0, \tag{1}$$

where x and $1-x$ represent the sold portion and unsold portion of the potential market (i.e., a fixable pool of consumers), respectively; u represents a firm's advertising budget.

The GVW model contains four parameters, i.e., the ad response index ρ, the decay index δ, the ad elasticity index α and the WoM index β. The former two parameters are original in the VW model: ρ represents the effectiveness of advertising effort, i.e., the response to advertising that acts positively on the unsold market share, and δ describes the loss of customers probably due to forgetting and competition that acts negatively on the sold portion of the market.

The latter two are the newly added parameters in the GVW model. The ad elasticity (α) is represented as the percentage change in advertising effort to the change of one percent in advertising budget, i.e., $\left(\Delta u^{\alpha}/u^{\alpha}\right)/(\Delta u/u)$, and u^{α} denotes the advertising effort. Essentially, the advertising effort can be explained as an effective part of advertising budget. In other words, not the entire budget, but this part can produce positive advertising effects (e.g., clicks, sales). In this sense, a higher ad elasticity index (α) implies that a unit advertising budget works more effectively. The ad elasticity index is normally fixed as a constant, i.e., $\alpha=1$ in the VW model and $\alpha=1/2$ in its derivatives (e.g., [3, 5, 11, 72]). Interestingly, doing so makes optimal budget allocation problems analytically solvable. In contrast to $\alpha=1$, $\alpha=1/2$ adds a favorable analytical property to the response model, i.e., the strict concavity [12], which captures the phenomenon of diminishing returns in advertising [31, 32]. However, the constant assumption of the ad elasticity index is unrealistic in practice. For example, Erickson [14] empirically found its value ranges from 0.116 to 0.726, and Yang et al. [73] reported the estimates ranging from 0.422 to 0.801. Thereby, the ad elasticity should be treated as a parameter to estimate, rather than a constant. Generally, the value of the ad elasticity index ranges between 0 and 1.0.

The WoM effect represents an additional process of WoM communications between the sold portion and the unsold portion of the potential market. That is, the WoM index (β) describes the level of communications between the two portions. More specifically, a smaller value of β means a higher WoM effect (i.e., $(1-\beta)$). The WoM effect is a type of additional advertising effects. Advertising campaigns may generate different types of WoM conversions [22], and in turn lead to various performance measures such as sales and stock returns [1, 58]. Both advertising and WoM communications can serve as product quality signals delivered to potential consumers [33]. Zhang et al. [80] investigated the effect of advertising and WoM in the context of movie diffusion process, and found that advertising has a direct impact on innovators fully mediated by the WoM effect. The initial introduction of WoM effect in the GVW model can be traced back to research by Sethi [64] and Sorger [66] where a square-root form of the untapped potential $\sqrt{1-x}$ (i.e., $\beta=1/2$) was used to capture the possible WoM effect. However, the assumption of the square-root form is hardly satisfactory because the WoM effect differs in different application domains [42]. Thus, the GVW model treats the WoM effect as a parameter to estimate, rather than a constant. Generally, the value of the WoM index ranges between 0 and 1.0.

The VW model and its derivatives can be viewed as special cases of the GVW model with different parameter settings. More specifically, the VW model is the

GVW with $\alpha = \beta = 1$; the Case model is the GVW with $\alpha = 1/2$ and $\beta = 1$ [2, 11], the Sethi-Sorger model is the GVW with $\alpha = 1$ and $\beta = 1/2$ [64, 66]; and the Chintagunta-Jain model is the GVW with $\alpha = \beta = 1/2$ [4]. We will explore it in more detail in Sect. 5.2.

3.2 Objective function

Consider the standard discounted profit obtained from advertising campaigns, the objective function is given as

$$J = \int_0^T (cx - u)e^{-rt}dt, \tag{2}$$

where c represents the gross profit rate of an advertiser obtained from a unit of market share, then $cx - u$ is the profit obtained from advertising campaigns at time t, and e^{-rt} is the discount factor.

3.3 Budget constraint

Let B denote the total advertising budget for the firm. Then the present value of the total advertising budget under a finite time horizon should not exceed B, which is given as

$$\int_0^T e^{-rt}udt \leq B. \tag{3}$$

In summary, Eqs. (1–3) constitute an optimal control model of advertising budget allocation (i.e., the GVW-OB model), where u and x are the decision variable and the state variable, respectively. We will derive the optimal budget strategy u^* allocated to the market by solving this model in Sect. 4.2.

4 Properties and solution

In this section, we study theoretical properties for our budget allocation model (Eqs. 1–3) and develop a solution process for situations with and without budget constraints.

4.1 Theoretical properties

First, we discuss the situation where an advertiser has a sufficient budget, which is given as

$$\max\left\{J = \int_0^T (cx - u)e^{-rt}dt\right\} \\ \dot{x} = \rho u^{\alpha}(1 - x)^{\beta} - \delta x, x(0) = x_0 . \tag{4}$$

By introducing the co-state variable λ (i.e., the shadow price of the market share), we can define the current-value Hamiltonian function as follows.

$$H = cx - u + \lambda\left[\rho u^{\alpha}(1 - x)^{\beta} - \delta x\right] \tag{5}$$

From the current-value Hamiltonian function (5), we can derive the co-state equation, which is given as

$$\dot{\lambda} = r\lambda - H_x = (r + \delta)\lambda - c + \rho\beta\lambda u^{\alpha}(1 - x)^{\beta - 1}, \quad \lambda(T) = 0. \tag{6}$$

We can obtain the first-order and the second-order derivatives of Hamiltonian function (5) with respect to the decision variable u, which are given as

$$H_u = -1 + \rho\alpha\lambda u^{\alpha-1}(1 - x)^{\beta} \\ H_{uu} = \rho\alpha(\alpha - 1)\lambda u^{\alpha-2}(1 - x)^{\beta} . \tag{7}$$

Suppose that there exists an interior maximum given the state and co-state variables (i.e., x and λ). Then let $H_u = 0$, we can obtain the optimal budget strategy

$$u^* = \left[\rho\alpha\lambda(1 - x)^{\beta}\right]^{\frac{1}{1-\alpha}}. \tag{8}$$

From Eq. (8), on the one hand, we can notice a property concerning the relationship between advertising budget (u) and the sold market share (x) where consumers are buying an advertiser's products or services, as described in Proposition 1.

Proposition 1 *The ad elasticity with respect to the sold market share is nonlinear and dynamic, given as*

$$\frac{\partial u}{\partial x} \cdot \frac{x}{u} = \frac{-\beta x}{(1 - \alpha)(1 - x)}. \tag{9}$$

Proposition 1 implies that, for an advertiser, the optimal advertising budget increasingly declines (or rises) as her market share expands (or shrinks) over time, with all else remaining unchanged. Essentially, the feature of dynamics concerning the relationship between the optimal budget and the sold market share can be explained by the basic principle of the VW model. That is, advertising budget acts on the unsold portion of the potential market; thus, the optimal budget is negatively related to the sold portion. The nonlinearity feature results from the exponent (i.e., the ad elasticity index α) to the decision variable (i.e., advertising budget u). This property could be regarded as a basic rule for the optimal advertising budget allocation.

On the other hand, Eq. (8) also reveals that the optimal path of advertising budget depends on both the state variable (x) and the co-state variable (λ). This property suggests examining dynamical behaviors of x and λ in a two-dimensional space S(λ,x), as illustrated in Figs. 1 and 2, where the curve defined by Eq. (1) ($\dot{x} = 0$) denoted by L_1, and the curve defined by Eq. (6) ($\dot{\lambda} = 0$) denoted by L_2, are contingent on the ad elasticity index (α) and the WoM index (β), as described below. The notations used in phase diagrams are listed in Table 1.

We can obtain phase diagrams of curves L_1 of λ ($\dot{x} = 0$) and L_2 of x ($\dot{\lambda} = 0$) in the feasible regions for seven cases. Please see Appendix A for more details. Figure 1 shows (x,λ) phase diagrams with three combinations when $0 < \alpha \le 1/2$, and Fig. 2 shows four combinations when $1/2 < \alpha < 1$.

As shown in Fig. 1, under the condition that the ad elasticity is relatively small (i.e., $0 < \alpha \le 1/2$), the co-state curve L_1 is convex and strictly monotonically increasing in x, and the shape of L_2 depends on the relationship between α and β: if $\beta = 1 - \alpha$ (Case-1), L_2 is a straight line parallel to the vertical axis; if $0 < \beta < 1 - \alpha$ (Case-2), L_2 is concave and strictly monotonically decreasing in λ; and if $1 - \alpha < \beta \le 2(1 - \alpha)$ (Case-3), L_2 is concave and strictly monotonically increasing in λ.

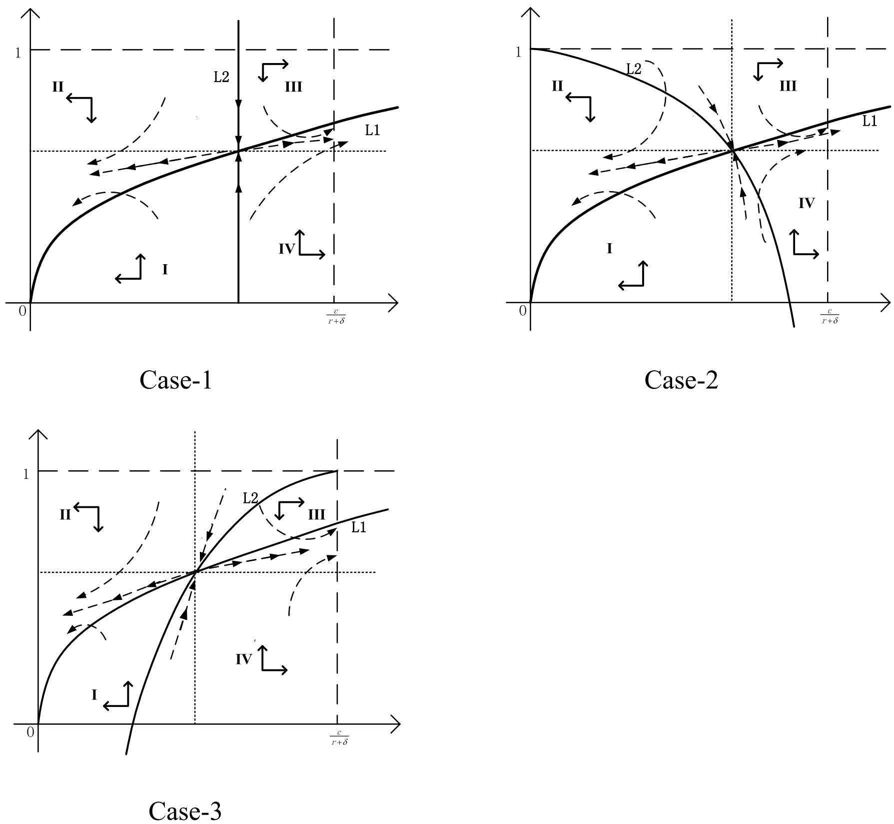


Fig. 1 (x, λ) phase diagrams with three combinations ($0 < \alpha \le 1/2$)

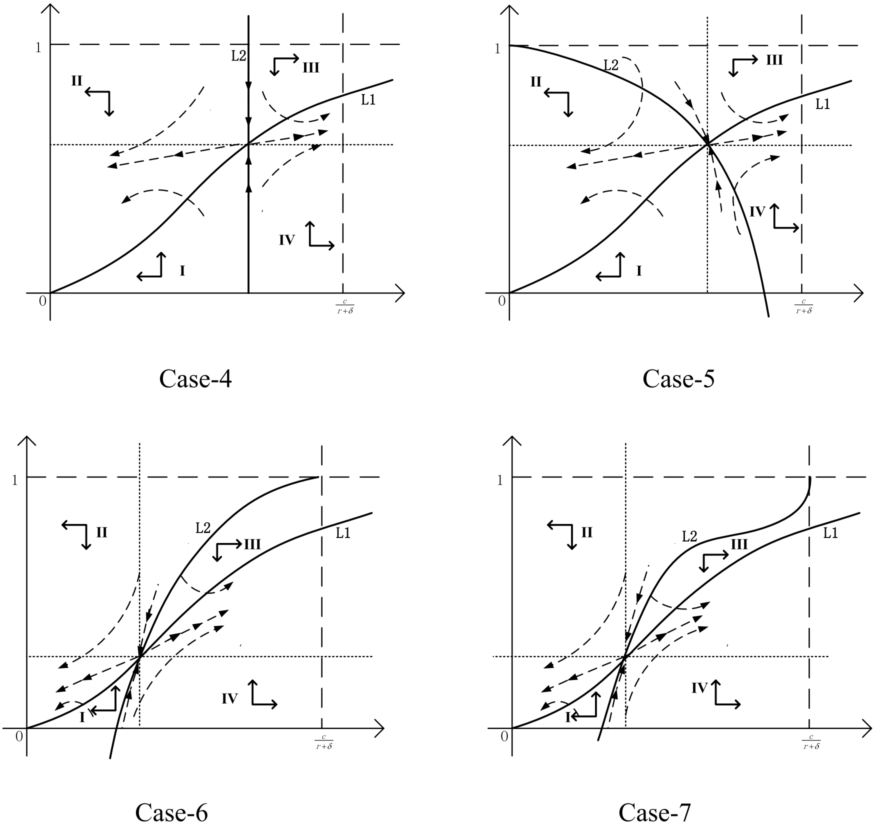


Fig. 2 (x,λ) phase diagrams with four combinations $(1/2 < \alpha < 1)$

Table 1 List of notations

Notation	Definition
I,II, III, IV	The four regions: Region I ($\dot{x} > 0$ and $\dot{\lambda} < 0$), Region II ($\dot{x} < 0$ and $\dot{\lambda} < 0$), Region III ($\dot{x} < 0$ and $\dot{\lambda} > 0$), and Region IV ($\dot{x} > 0$ and $\dot{\lambda} > 0$)
L_1	The curve defined by Eq. (1) with $\dot{x} = 0$
L2	The curve defined by Eq. (6) with $\dot{\lambda} = 0$
Dashed line – – –	The dashed line parallel to the vertical axis is with $x = 1$; and the dashed line parallel to the horizontal axis is with $\lambda = \frac{c}{r+\delta}$
Dotted line ……	The dotted line parallel to the vertical axis is with $x = \overline{x}$; the dotted line parallel to the horizontal axis is with $\lambda = \overline{\lambda}$; and $(\overline{\lambda}, \overline{x})$ is the steady-state point
Right arrow →	The increasing of λ in a region
Left arrow ←	The decreasing of λ in a region
Up arrow ↑	The increasing of x in a region
Down arrow ↓	The decreasing of x in a region
Dashed line arrow – – –>	The trajectory of (λ,x) in the phase diagram

As shown in Fig. 2, under the condition that the ad elasticity is relatively large (i.e., $1/2 < \alpha < 1$), the co-state curve L_1 is concave-convex and strictly monotonically increasing in x, and likewise, the shape of L_2 depends on the relationship between α and β: if $\beta = 1 - \alpha$ (Case-4), L_2 is a straight line parallel to the vertical axis; if $0 < \beta < 1 - \alpha$ (Case-5), L_2 is concave and strictly monotonically decreasing in λ; if $1 - \alpha < \beta \le 2(1 - \alpha)$ (Case-6), L_2 is concave and strictly monotonically increasing in λ; if $2(1 - \alpha) < \beta$ (Case-7), L_2 is concave-convex and strictly monotonically increasing in λ.

As for the market share dynamics ($\dot{x}$), it reaches a steady-state equilibrium on L_1, i.e., $\dot{x} = 0$; and it is negative in the space over L_1 and positive under L_1. Similarly, as for the co-state dynamics ($\dot{\lambda}$), it reaches a steady-state equilibrium on L_2, i.e., $\dot{\lambda} = 0$; and it is negative in the left space of L_2, and positive in the right space. From Figs. 1 and 2, we can notice that L_1 and L_2 segments the $S(\lambda, x)$ space into four regions: Region I ($\dot{x} > 0$ and $\dot{\lambda} < 0$), Region II ($\dot{x} < 0$ and $\dot{\lambda} < 0$), Region III ($\dot{x} < 0$ and $\dot{\lambda} > 0$), Region IV ($\dot{x} > 0$ and $\dot{\lambda} > 0$). Moreover, for each case, there exists a unique intersection of L_1 and L_2 in the region $x \in [0, 1] \times \lambda \in \left[0, \frac{c}{r+\delta}\right]$, which is termed as the steady-state point, denoted by $(\overline{\lambda}, \overline{x})$. The existence and uniqueness of the steady-state point are described in the following proposition.

Proposition 2 *There exists one and only one steady-state point* $(\overline{\lambda}, \overline{x})$ *in the* $S(\lambda, x)$ *space, and the steady-state point is the saddle point.*

Proof See Appendix B. Graphically, in the (λ, x) phase diagrams (Figs. 1 and 2), there exists exactly two paths (one from each side of $\overline{\lambda}$ or $\overline{x}$) which tend toward the steady-state point and two paths moving away from it. The trajectory of optimal solution asymptotically approaches one of the diverging paths as $t \to T$. More specifically, for an advertiser with a sufficient budget, as time goes by, a) when $x_0 > \overline{x}$, the state variable x decreases and converges to the steady-state point, the co-state variable λ remains unchanged (i.e., $\lambda = \lambda_0$) (Case-1 and Case-4), or decreases (Case-3, Case-6 and Case-7), or increases (Case-2 and Case-5), eventually converges to the steady-state point; b) when $x_0 < \overline{x}$, the state variable x increases and converges to the steady-state point, the co-state variable λ remains unchanged (i.e., $\lambda = \lambda_0$) (Case-1 and Case-4), or decreases (Case-2 and Case-5), or increases (Case-3, Case-6, and Case-7), eventually converges to the steady-state point. □

Next, we discuss the situation where an advertiser has a limited budget, which is given as

$$\begin{aligned} &\max \left\{ J = \int_0^T (cx - u)e^{-rt}dt \right\} \\ &\dot{x} = \rho u^{\alpha}(1 - x)^{\beta} - \delta x, \ x(0) = x_0 \cdot \\ &\int_0^T e^{-rt}udt \le B \end{aligned} \tag{10}$$

Following Yang et al. [77], this optimal control problem can be transformed to an equivalent one with the budget constraint replaced by a new auxiliary state variable R(t), defined as the present value of the remaining advertising budget, i.e., $R(t) = R(T) + \int_t^T e^{-rs} u(s)ds$. Then we have $\dot{R} = -e^{-rt}u$. The resulting equivalent optimal control formulation is then given as follows.

$$\begin{aligned} &\max \left\{ J = \int_0^T (cx - u)e^{-rt}dt \right\} \\ &\dot{x} = \rho u^{\alpha}(1 - x)^{\beta} - \delta x \\ &\dot{R} = -e^{-rt}u \\ &u \geq 0, R(0) = B, R(T) \geq 0 \end{aligned} \tag{11}$$

By introducing the co-state variable λ and μ, we can define the current-value Hamiltonian functions as follows.

$$H = cx - u + \lambda\left[\rho u^{\alpha}\left(1 - x^{\beta}\right)\right] + \mu\left(-ue^{-rt}\right) \tag{12}$$

From the current-value Hamiltonian function (12), we can derive the co-state equations:

$$\dot{\lambda} = r\lambda - H_x = (r + \delta)\lambda - c + \rho\beta\lambda u^{\alpha}(1 - x)^{\beta-1}, \quad \lambda(T) = 0, \tag{13}$$

$$\dot{\mu} = r\mu, \mu(T) \geq 0, \quad \mu(T)R(T) = 0. \tag{14}$$

From Eq. (14) we obtain $\mu = C_0 e^{rt} \geq 0$. Under the transversality condition, $\mu(T) = C_0 e^{rT} \geq 0$. In the case without budget constraints, we have $\mu(T) = 0$, then $C_0 = 0$ and $\mu(t) = C_0 e^{rt} = 0$, $\forall t \geq 0$; while in the case with budget constraints, $\mu(T) > 0$, then $C_0 > 0$ and $\mu(t) = C_0 e^{rt} > 0$, $\forall t \geq 0$.

From Eq. (12) we have

$$H = cx - u + \lambda[\rho u^{\alpha}\left(1 - x\right)^{\beta}] - C_0 u.$$

Suppose that there exists an interior maximum given the state and co-state variables (i.e., x, λ and μ). Then let $H_u = 0$, we can obtain the optimal budget strategy

$$u^* = \left[\frac{\rho\alpha\lambda(1 - x)^{\beta}}{1 + C_0}\right]^{\frac{1}{1-\alpha}}. \tag{15}$$

In the following, we investigate dynamical behaviors of state and co-state variables in the (x, λ) phase diagrams. Similar to the situation where an advertiser has a sufficient budget, seven phase diagrams of curves L_1' of λ ($\dot{x} = 0$) and L_2' of x ($\dot{\lambda} = 0$) can be obtained. Please see Appendix C for more details. The property of the steady-state point in the situation with budget constraints is described in Proposition 3.

Proposition 3 *In the situation with budget constraints, a) there exists one and only one steady-state point* $\left(\overline{\lambda'}, \overline{x'}\right)$ *in* $S(\lambda, x)$*, and the steady-state point is the saddle* *point; b) compared to the situation without budget constraints, the steady-state point moves to right bottom, i.e.,* $\overline{x'} < \overline{x}$, $\overline{\lambda'} > \overline{\lambda}$.

Proof See Appendix C. In the situation with budget constraints, the state variable x and the co-state variable λ follow similar evolution patterns to the situation without budget constraints. □

4.2 A computational solution

In the following, we present a computational solution for optimal budget allocation in situations with (Model 4) and without (Model 10) budget constraints.

From Eq. (10), we have the following Hamilton–Jacobi equation

$$V_{\eta,t} + ce^{-rt}x + \left(\frac{1}{\alpha} - 1\right)\left[\frac{\alpha e^{\alpha rt}}{(1+\eta)^{\alpha}}\rho(1-x)^{\beta}V_{\eta,x}\right]^{\frac{1}{1-\alpha}} - \delta x V_{\eta,x} = 0. \tag{16}$$

The boundary condition $V_{\eta}(x, T) = 0$, and the optimal strategy is given as

$$u = \left[\frac{\rho\alpha V_{\eta,x}(1-x)^{\beta}e^{rt}}{1+\eta}\right]^{\frac{1}{1-\alpha}}. \tag{17}$$

For the case without budget constraints, let $\eta = 0$, we have the following Hamilton–Jacobi equation

$$V_t + ce^{-rt}x + \left(\frac{1}{\alpha} - 1\right)\left[\alpha e^{\alpha rt}\rho(1-x)^{\beta}V_x\right]^{\frac{1}{1-\alpha}} - \delta x V_x = 0, \quad V(x,T) = 0,$$

and the optimal strategy is given as

$$u = \left[\rho\alpha V_x(1-x)^{\beta}e^{rt}\right]^{\frac{1}{1-\alpha}}.$$

The optimal trajectory of budget for the situation where an advertiser has a sufficient budget, i.e., u_0^*, represents the solution for Model (4), which can be directly derived when $\eta = 0$; while in the situation where an advertiser has a limited budget, the optimal budget strategy (u_η^*) for Model (10) can be obtained by determining the Lagrange multiplier ($\eta > 0$) in an iterative way. The detailed steps of the numerical procedure for finding the optimal budget allocation strategy and corresponding market share are specified in Algorithm 1.

This procedure includes two sub-procedures [16]. The first sub-procedure calculates the value function and the allocated budget for all possible combinations of market share x and time t. The second is intended to determine the optimal budget strategy and corresponding market share sequentially. The results from the backward sweep two matrices of the optimal value function $V_\eta(x, t)$ and the optimal $u_\eta(t)$ for each starting time $t = \{0, \Delta t \cdot 1, \Delta t \cdot 2, \ldots, T\}$ with the initial cumulative market share $x = \left\{x_0, x_0 + \Delta x \cdot 1, x_0 + \Delta x \cdot 2, \ldots, 1\right\}$. This procedure involves the choice of the Lagrange multiplier increment ($\Delta\eta$), the time interval (Δt), and the market share increment (Δx).

Algorithm 1. (Two-Stage Optimal budget allocation)
Input: the initial market share x_0, the total budget B , and the planning time horizon T.
Output: the optimal budget strategy $u_\eta(t)$ and the corresponding market share $x(t)$.

Procedure

Step 1: Let $t = T; x = \{x_0, x_1, \cdots, x_k, \cdots 1\}$, where $x_{k+1} = x_k + \Delta x$. Initialize $V_\eta(x, T) = 0, \ \forall x$.

Step 2: Let $\eta = 0$.

Sub-procedure 1: Backward-Sweep

Step 3: Approximate $V_{\eta,x}(x, t)$ for all x.

Step 4: Calculate $V_{\eta,x}(x, t)$ according to Equation (16).

Step 5: Calculate $u_\eta(x, t)$ according to Equation (17).

Step 6: Let $t = t - \Delta t$. Update $V_\eta(x, t - \Delta t) = V_\eta(x, t) - V_{\eta,t}\Delta t$.

Step 7: If $t = 0$, then stop and continue to Sub-procedure 2; otherwise, go to Step 3.

End Sub-procedure 1

Sub-procedure 2: Forward Sweep

Step 8: Let $t = t_0$, determine the optimal $u_\eta(t)$ based on the $x(t)$.

Step 9: Calculate Δx based on $u_\eta(t)$, and let $x(t+1) = x(t) + \Delta x$. $x(t+1)$ lies in $[x_k, x_{k+1}]$ for a certain k, which is determined through the interpolation.

Step 10: Let $t = t + 1$, and repeat Steps 8-9 until the final planning horizon (i.e., T) is reached.

Step 11: Calculate the total advertising budget $\overline{B} = \int_0^T e^{-rt} u_\eta(t) dt$, if $\overline{B} \le B$, then continue to Step 12; otherwise, let $\eta = \eta + \Delta\eta$, go to Step 3.

End Sub-procedure 2

Step 12: Return the optimal budget strategy $u_\eta(t)$ and the corresponding market share $x(t)$.

End Procedure

5 Experimental validation

In this section, we design computational experiments to validate the proposed GVW-OB strategy and its identified properties.

Our experimental evaluation serves the following twofold purposes. First, we are intended to evaluate the effectiveness of the GVW-OB strategy by comparing it with four baselines in terms of payoff and return on investment (ROI). Each baseline corresponds to an optimal budget allocation strategy based on a VW-type advertising response model (i.e., the VW model and its derivatives). Second, we attempt to prove desirable properties of our GVW-OB model as discussed in Sect. 4 by conducting sensitivity analysis. Specifically, we examine influences of the ad elasticity index and the WoM index on the performance of our GVW-OB strategy. Next, we provide details about our experimental setup and key results.

5.1 Data description and experimental setting

We collected three real-world datasets from advertising campaigns by a large U.S. e-commerce retailer, a European e-commerce retailer and a Chinese e-commerce retailer, respectively. The first dataset records advertising campaigns on Google AdWords spanning four calendar years, which contains approximately seven million records from almost 55,000 advertisements. Each record in the first dataset includes the key phrase that triggered the ad, number of impressions, number of clicks, the average click-through rate (CTR), the average cost-per-click (CPC), the number of conversions (or orders), the total sales revenues, and the total number of items ordered. The second dataset records historical information of advertising campaigns on Facebook Ads. The European e-commerce retailer operates in eight countries and the data is from the Finnish branch. The dataset contains 62,802 records from 95 advertisements during a 20-month period. The third dataset records advertising campaigns on Baidu which is the most popular search engine in China. The dataset contains 34,019 records from 24 advertisements during an 8-month period. From these datasets, we collect information about the advertisers' total advertising budget and budget decisions, impressions, and clicks generated from their advertising campaigns, the average CTR, the average CPC, conversions, and so on. Finally, we also generate data from historical advertising reports to support computational experiments to verify properties of our GVW-OB strategy.

Following the extant literature [40, 69], advertising responses to advertising budget can be either concave (e.g., [6, 18]) or S-shaped (e.g., [9, 17]). Through examining relationships between advertising responses (i.e., clicks) and advertising budget based on our datasets, we found empirical evidences for concave and S-shaped curves, respectively, as shown in Fig. 3. Thus, we also attempt to examine whether our GVW-OB strategy and four baselines perform differently on the two settings exhibiting concave and S-shaped advertising responses, respectively.

In the experiments for comparisons, we applied a 2×3 design with two shapes of advertising responses (i.e., concave and S-shaped) and three datasets (i.e., a U.S. firm's advertising campaigns on Google AdWords, a European firm's advertising campaigns on Facebook Ads and a Chinese firm's advertising campaigns on Baidu Ads) to evaluate the performance of our GVW-OB strategy and four baselines. The two widely used criteria, i.e., payoff and ROI, were taken to measure the performance of these budget allocation strategies.

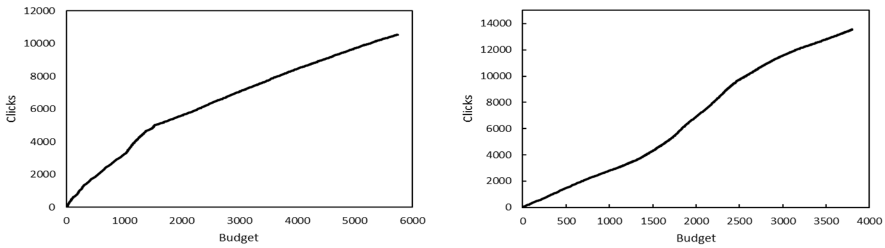


Fig. 3 The shape of responses to advertising budget

5.2 Comparisons

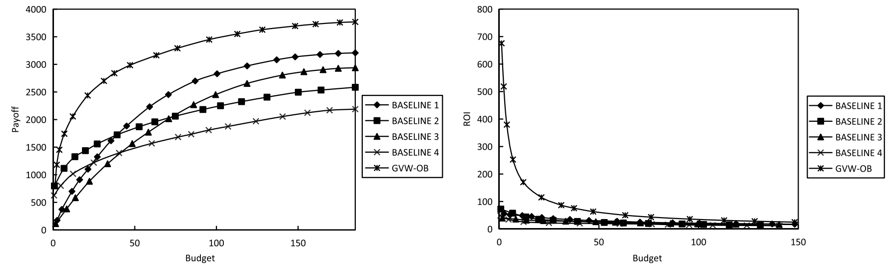
In this section, we compare our GVW-OB strategy with four baselines. The first is referred to as the optimal budget allocation strategy based on the VW model ($\alpha = \beta = 1$) (called BASELINE1), as proposed by Sethi [63]. The other three baselines are optimal budget strategies based on derivatives of the VW model, with different parameter settings. The second baseline is referred to as the optimal budget allocation strategy based on the VW derivative with $\alpha = 1/2$ and $\beta = 1$ called BASELINE2, which has been investigated in [2, 5, 59]. The third baseline is referred to as the optimal budget allocation strategy based on the VW derivative with $\alpha = 1$ and $\beta = 1/2$ called BASELINE3, which has been explored in [64, 66, 77]. The fourth baseline is referred to as the optimal budget allocation strategy based on the VW derivative with $\alpha = \beta = 1/2$ called BASELINE4, derived from [4, 13, 14, 51].

The deep neural network (DNN)-based method developed by Yang et al. [73] is used to learn modeling parameters of the GVW model and other four advertising response models of those baselines based on our datasets. More specifically, for the VW model and its derivatives, as the two parameters (i.e., the ad elasticity index and the WoM index) are given a prior as constants, the other two (i.e., the ad response index and the decay index) are estimated; while for the GVW model, we need to estimate all the four modeling parameters.

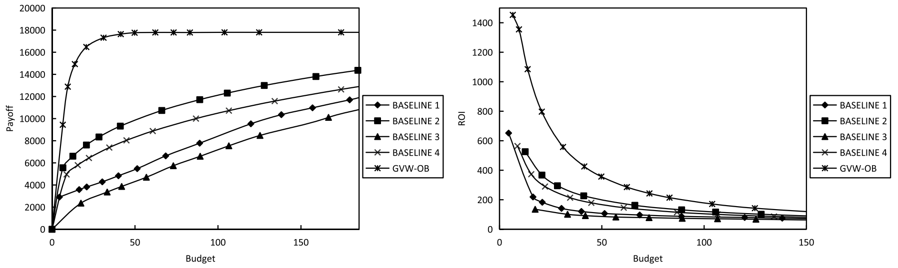
We execute two sets of experiments on each dataset: one on the data exhibiting concave advertising responses, and the other on the data favoring S-shaped responses. Figure 4a, b shows the payoff and ROI obtained by the GVW-OB strategy and four baselines at different (total) budget levels (B) in the two experimental settings on the Google dataset, respectively. Similarly, Fig. 5a, b presents the results on the Facebook dataset, and Fig. 6a, b illustrates the results on the Baidu dataset.

From Figs. 4a, b, 5a, b, 6a, b, we can draw the following conclusions.

First, in both concave and S-shaped settings on the three datasets, the GVW-OB strategy outperforms the four baseline strategies in terms of both payoff and ROI. This phenomenon can be explained in two aspects. On the one hand, the GVW-OB strategy is built based on the GVW model, which allows deriving four parameters from the data, rather than assuming parameters to be constants or in some specific function forms. The better data suitability helps improve the performance of the GVW-OB strategy. On the other hand, our GVW-OB strategy is a closed-loop



a Payoff and ROI of the GVW-OB Strategy and Four Baselines on the Google Dataset (concave)



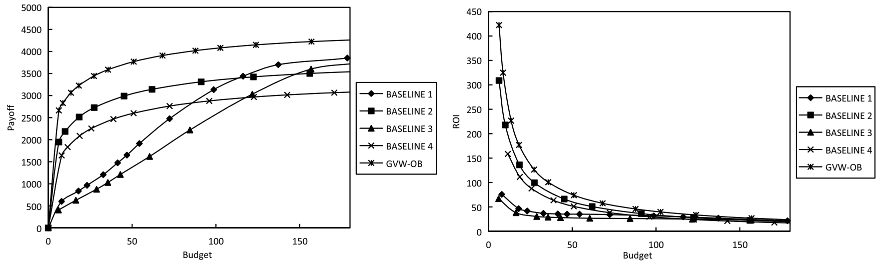
b Payoff and ROI of the GVW-OB Strategy and Four Baselines on the Google Dataset (S-shaped)

Fig. 4 **a** Payoff and ROI of the GVW-OB strategy and four baselines on the Google Dataset (concave). **b** Payoff and ROI of the GVW-OB strategy and four baselines on the Google Dataset (S-shaped)

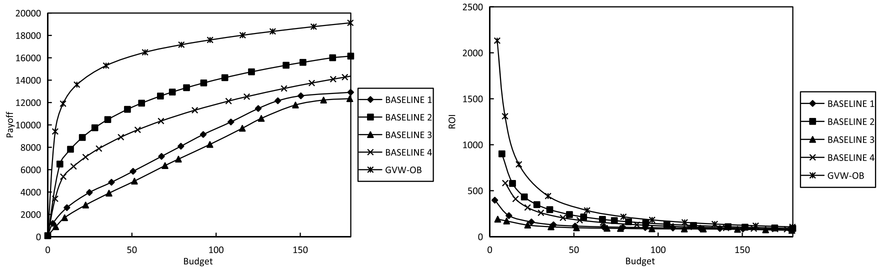
policy in that it observes the state variable (i.e., market share) and updates the optimal strategy continuously by tracking the advertising performance. Therefore, the GVW-OB strategy can better fit practical advertising situations, thus achieves the best performance in terms of payoff and ROI.

Second, the four baselines perform differently in concave and S-shaped settings. Specifically, on the three datasets, the order in the obtained payoff on the concave setting is BASELINE1, BASELINE3, BASELINE2, and then BASELINE4, while the order in the payoff on the S-shaped setting is BASELINE2, BASELINE4, BASELINE1, and then BASELINE3. Moreover, this is consistent with the performance order of the four baselines in terms of ROI. This finding suggests that BASELINE1 and BASELINE3 favor concave advertising responses, while BASELINE2 and BASELINE4 support S-shaped responses. The possible reason for this phenomenon is that, compared to linear models, nonlinear advertising models might better tally with the situation with S-shaped responses. That is, the nonlinearity modeling feature with respect to advertising budget and the untapped market share improves the adaptability of budget allocation strategies to handling complex advertising situations.

Next, we investigate the optimal budget path and the evolution of market share over time. In the experiments, the total budget is set as $B = 130$; the initial market share is set as $x_0 = 0.0$. Figure 7a, b shows the optimal budget path and the evolution of market share over time obtained by the GVW-OB strategy and four baselines in



a Payoff and ROI of the GVW-OB Strategy and Four Baselines on the Facebook Dataset (concave)



b Payoff and ROI of the GVW-OB Strategy and Four Baselines on the Facebook Dataset (S-shaped)

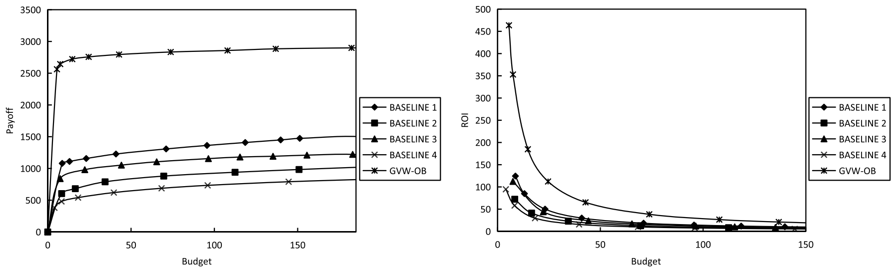
Fig. 5 **a** Payoff and ROI of the GVW-OB strategy and four baselines on the Facebook Dataset (concave). **b** Payoff and ROI of the GVW-OB strategy and four baselines on the Facebook Dataset (S-shaped)

the two experimental settings on the Google dataset, respectively. Similarly, Fig. 8a, b presents the results on the Facebook dataset, and Fig. 9a, b illustrates the results on the Baidu dataset.

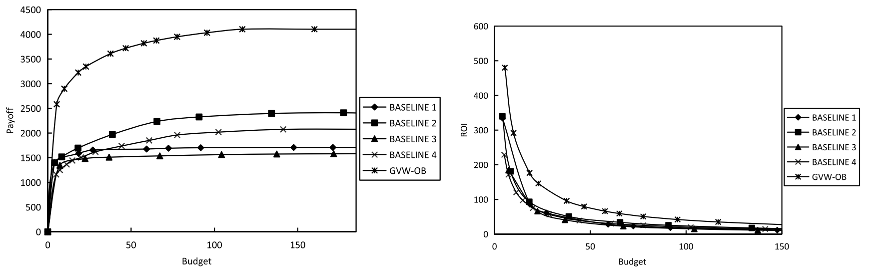
From Figs. 7a, b, 8a, b, 9a, b, we make the following observations.

First, in both concave and S-shaped settings on the three datasets, BASELINE1 and BASELINE3 invest a large amount of budget and obtain a large market share at the initial stage, then slash the budget, which results in the decreasing of market share. The possible reason for this phenomenon is that the advertising effort in the underlying response models for these two baselines is linear in the budget. This linearity assumption takes each unit of budget equally, which is barely satisfactory in practice. On the contrary, the nonlinearity for the GVW-OB strategy, BASELINE2, and BASELINE4 entitles them to the effect of diminishing returns in advertising.

Second, in both concave and S-shaped settings on the three datasets, the GVW-OB strategy, BASELINE2 and BASELINE4 show a similar pattern with respect to the evolution of market share. However, the GVW-OB strategy obtains a higher level of market share than BASELINE2 and BASELINE4. This indicates that, under the same budget constraint, the GVW-OB strategy could grab more market share for advertisers. Moreover, optimal path of the budget over time by BASELINE2 and BASELINE4 are different from that by the GVW-OB strategy.



a Payoff and ROI of the GVW-OB Strategy and Four Baselines on the Baidu Dataset (concave)



b Payoff and ROI of the GVW-OB Strategy and Four Baselines on the Baidu Dataset (S-shaped)

Fig. 6 **a** Payoff and ROI of the GVW-OB strategy and four baselines on the Baidu Dataset (concave). **b** Payoff and ROI of the GVW-OB strategy and four baselines on the Baidu Dataset (S-shaped)

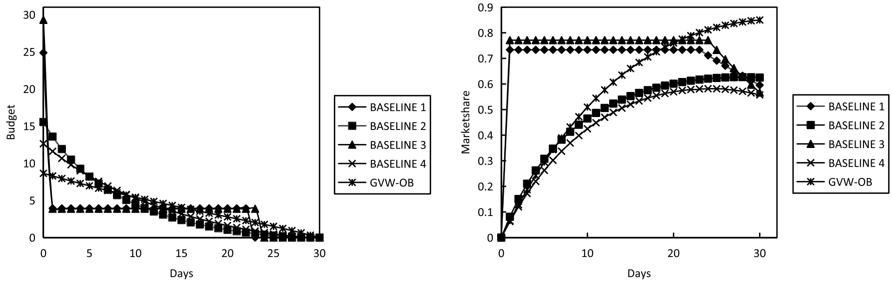
5.3 Sensitivity analysis

In the following, we design computational experiments to conduct sensitivity analysis for the GVW-OB strategy with respect to its modeling parameters. In particular, we focus on the two distinct indexes for the GVW model that are distinguishable from other VW-type models, namely the ad elasticity index and the WoM index. For each modeling parameter, we first investigate its effect on payoff and ROI obtained by the GVW-OB strategy at different budget levels, then turn to its effect on the optimal budget path and the evolution of market share over time. The datasets used in the following experiments were generated from historical advertising logs.

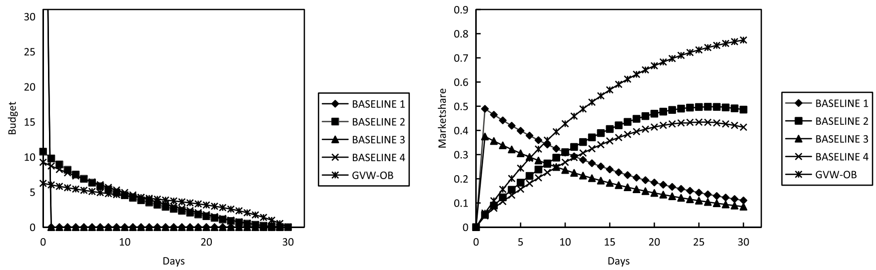
5.3.1 Influence of the ad elasticity index

We first investigate the influence of the ad elasticity index (α) on the performance of the GVW-OB strategy. Figure 10 illustrates payoff and ROI obtained by the GVW-OB strategy at different (total) budget levels (B) with different ad elasticity indexes.

From Fig. 10, we can notice the following observations. First, comparing settings with different ad elasticity indexes, we can clearly see that a larger ad elasticity leads to a higher performance in terms of both payoff and ROI. Second, the payoff increases monotonically with the total budget, but the ROI decreases. Specifically, the payoff grows steadily until reaching the budget cap where the marginal payoff



a Optimal Budget Path and Evolution of Market Share over Time on the Google Dataset (concave)



b Optimal Budget Path and Evolution of Market Share over Time on the Google Dataset (S-shaped)

Fig. 7 **a** Optimal budget path and evolution of market share over time on the Google Dataset (concave). **b** Optimal budget path and evolution of market share over time on the Google Dataset (S-shaped)

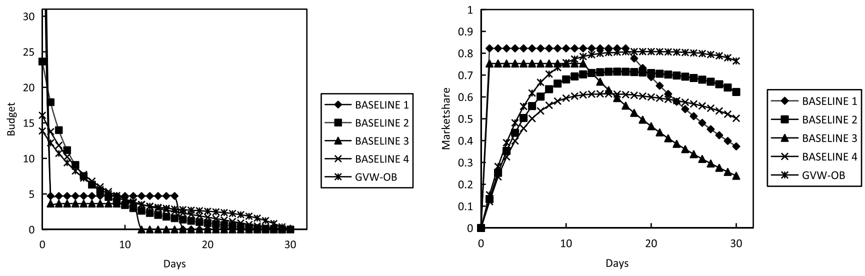
becomes zero; meanwhile, the ROI shrinks and then stays at a relatively steady level. This phenomenon reflects the law of diminishing marginal utility.

Next, we examine the effect of the ad elasticity index on the optimal budget path and the evolution of market share over time in the case with a sufficient budget. In the experiments, the initial market share is set as $x_0 = 0.0$. Figure 11 presents the optimal budget path and the evolution of market share over time, with different ad elasticity indexes.

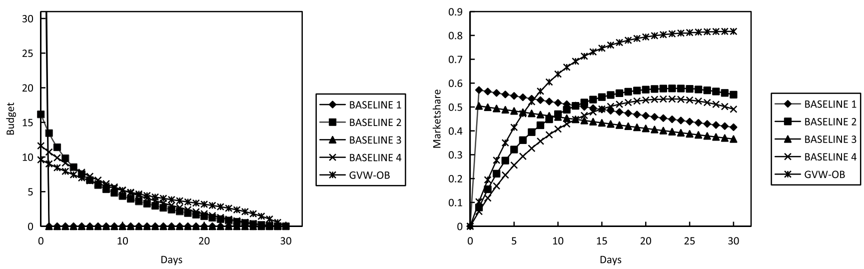
As shown in Fig. 11, a larger ad elasticity index empowers higher levels of optimal budget and market share. In addition, except for the setting with $\alpha = 1$, the optimal budget decreases over time, while the corresponding market share initially increases and then exhibits a decreasing trend. Note that, in our experiments, in settings with smaller ad elasticities (i.e., $\alpha = 0.1$ and $\alpha = 0.3$), the evolution of market share does not show the decreasing phase possibly because of a finite time horizon (i.e., $T = 30$).

5.3.2 Influence of the WoM index

In the following, we investigate the influence of the WoM index (β) on the performance of the GVW-OB strategy. Figure 12 illustrates the payoff and ROI obtained



a Optimal Budget Path and Evolution of Market Share over Time on the Facebook Dataset (concave)



b Optimal Budget Path and Evolution of Market Share over Time on the Facebook Dataset (S-shaped)

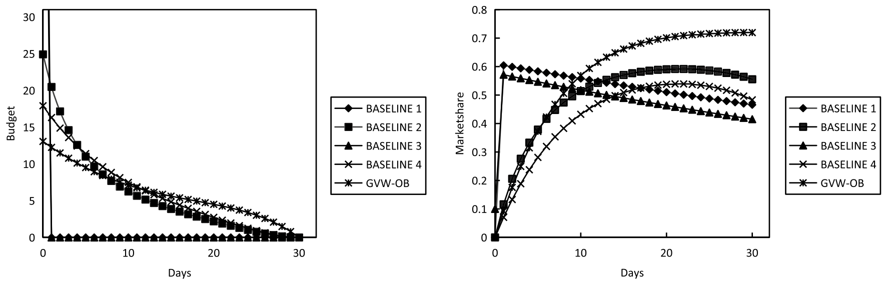
Fig. 8 **a** Optimal budget path and evolution of market share over time on the Facebook Dataset (concave). **b** Optimal budget path and evolution of market share over time on the Facebook Dataset (S-shaped)

by the GVW-OB strategy at different (total) budget levels (B) with different WoM indexes.

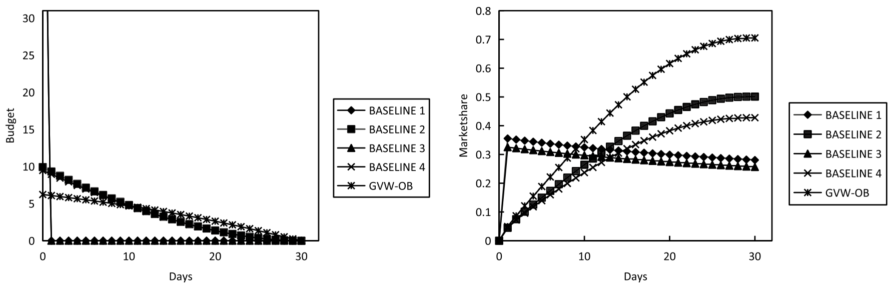
From Fig. 12, we can see that, a smaller WoM index, which implies a higher WoM effect, results in higher payoff and ROI. This is because a higher WoM effect means a higher level of communications between the sold portion and the unsold portion of the potential market, which enlarges the advertising effectiveness from each unit of the budget. Moreover, similarly, the payoff monotonically increases with the total budget until reaching the budget cap, but the ROI decreases and then stays at a relatively steady level.

Next, we examine the effect of the WoM index on the optimal budget path and the evolution of market share by the GVW-OB strategy, in the case with a sufficient budget. Similarly, in the experiments, the initial market share is set as $x_0 = 0.0$. Figure 13 presents the optimal budget path and the evolution of market share over time, with different WoM indexes.

Figure 13 reveals that, in the situation with higher WoM effects (i.e., smaller values of β), the optimal budget and corresponding market share are higher. The WoM effect can be considered as the additional effectiveness obtained from advertising campaigns in that it enlarges the advertising coverage. As reported



a Optimal Budget Path and Evolution of Market Share over Time on the Baidu Dataset (concave)



b Optimal Budget Path and Evolution of Market Share over Time on the Baidu Dataset (S-shaped)

Fig. 9 **a** Optimal budget path and evolution of market share over time on the Baidu Dataset (concave). **b** Optimal budget path and evolution of market share over time on the Baidu Dataset (S-shaped)

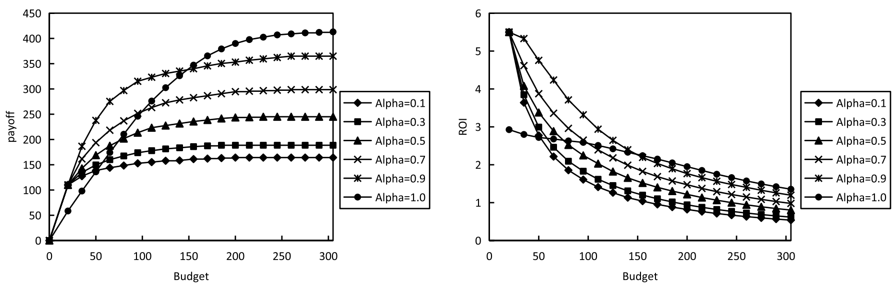


Fig. 10 Payoff and ROI at different budget levels with different ad elasticity indexes

by prior studies, advertising campaigns could raise the WoM effect, which in turn leads to better advertising performance [22, 58]. In other words, a higher WoM effect improves the advertising efficiency, which encourages advertisers to invest more budget and in turn get more market share. In addition, during the promotion period, the optimal budget decreases and corresponding market share increases over time.

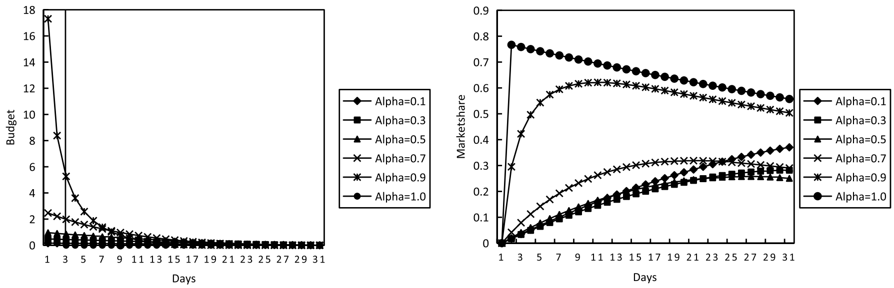


Fig. 11 Optimal budget path and evolution of market share over time with different ad elasticity indexes

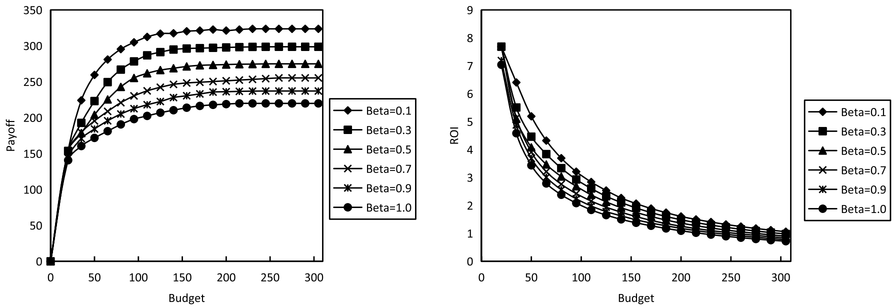


Fig. 12 Payoff and ROI at different budget levels with different WoM indexes

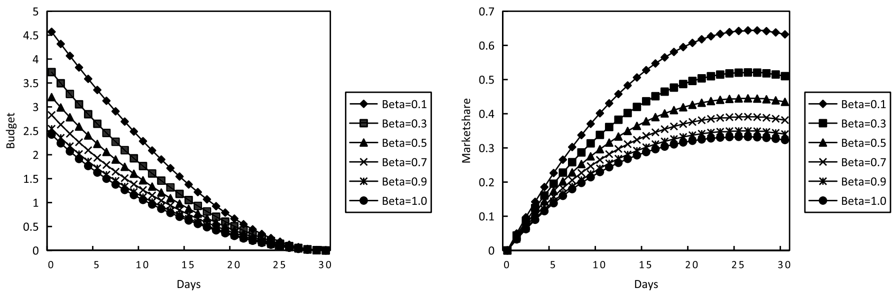


Fig. 13 Optimal budget path and evolution of market share over time with different WoM indexes

6 Discussion

This research generates several valuable managerial insights for advertisers to make budget allocation decisions. First, the GVW-OB strategy performs better in terms of both payoff and ROI. Moreover, the GVW-OB strategy could help advertisers obtain a larger market share from a certain amount of budget. Essentially, budget allocation is a multi-faceted task, which calls for a comprehensive examination of all four modeling indexes to obtain the most suitable solution to practical advertising campaigns. Of most importance, the nonlinearity feature with respect to advertising budget (raised by the ad elasticity index) and the unsold market (raised by the WoM index) enhances the adaptability of the GVW-OB strategy to handle budget allocation problems in complex advertising situations (e.g., the situation exhibiting the S-shaped rather than concave advertising responses), especially in the case when an advertiser has a limited amount of budget. In addition, advertising revenues play an important role in supporting access to many free services provided by digital platforms, which have rapidly become essential to people [30]. Thus, it is necessary for digital media providers to develop tools for advertisers to assist budget decisions in such complicated environments.

Second, a larger ad elasticity empowers higher levels of the optimal budget and corresponding market share, thus leading to higher payoff and ROI. This finding is consistent with that reported by Yang et al. [77]. Advertisers with higher ad elasticity are recommended to invest more budget in order to maximize their expected returns.

Third, a larger WoM effect leads to higher levels of the optimal budget and corresponding market share, in turn resulting in a higher payoff and ROI. That is, for advertisers, a high WoM effect could help advertisers earn more advertising returns from each penny. This suggests that advertisers should encapsulate social elements in their advertising campaigns in order to create WoM effects in the potential market. As for media providers, they are encouraged to design socially enriched mechanisms to facilitate conversions among potential consumers, especially in media forms without social privilege of birth (e.g., search advertising), and provide advertisers a richer set of metrics related to the WoM effect.

Finally, from a methodological perspective, without loss of generality, our GVW-OB strategy provides a feasible solution for advertisers to dynamically make optimal budget allocation, which can be easily applied to a variety of advertising media.

However, we also realize several shortcomings of our work. First, in this study, we focus on the optimal budget allocation problem using the GVW model as its advertising dynamics under a finite time horizon. We mathematically analyzed properties of our GVW-OB model and derived a feasible solution, while emphasizing roles of the ad elasticity index and the WoM index in budget allocation decisions. However, the underlying mechanism behind advertising response processes is still to be examined. Second, new advertising forms that have emerged in recent years (e.g., search advertising) contain some distinctive features and operational rules, compared to traditional advertising. Thus, it urgently calls for deep-level research efforts to adapt the GVW-OB strategy specifically to new advertising media, by developing and encapsulating some additional indexes representing inherent features. Third, this study explores budget allocation decisions in the situation with a single advertiser. As noted, the situation with two or more advertisers leads to more complicated strategies.

7 Conclusions and future directions

In this research, we present an optimal budget allocation strategy called the GVW-OB using the GVW model as its advertising dynamics under a finite time horizon. Moreover, we discuss some desirable properties and develop a feasible solution to our budget allocation model. Furthermore, computational experiments are conducted to evaluate our GVW-OB model and its identified properties. Experimental results show that our GVW-OB strategy outperforms four baselines in terms of both payoff and ROI in either concave or S-shaped settings; the ad elasticity index and the WoM index have significant influences on the performance of the GVW-OB strategy; and the nonlinearity modeling feature raised by the two indexes improves the adaptability of the GVW-OB strategy to handling complex advertising situations.

In this direction, several interesting perspectives deserve further research efforts. First, we plan to conduct empirical studies applying the GVW model in various advertising media. In order to obtain unbiased estimations, it is suggested to control for multi-market effect decomposing the global and local components of advertising and the synergy effect isolating the direct impact of advertising variables analyzed. In addition, we also intend to empirically explore the functional shape of advertising responses and possible conditions for the inflection point and the threshold effect that would help develop sophisticated advertising models. Another interesting direction is to adapt the proposed GVW-OB strategy in uncertain advertising environments. Last but not least, it would also be interesting to explore multi-channel budget allocation decisions by encoding the potential heterogeneity among different media vehicles.

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